CASE 6 $\qquad f = x^m$

$$s_f = \sqrt{\left(\frac{\partial x^m}{\partial x}\right)^2 (s_x)^2} = \sqrt{m\left(x^{m-1}\right)^2 (s_x)^2} =$$

$$= m \frac{x^m}{x} s_x$$

$$\frac{s_f}{f} = \frac{m \cdot s_x}{x}$$

CASE 4 etc MORE CASES

$$f = e^{x} \qquad (S_x = dx)$$

$$S_f = \sqrt{\left(\frac{\partial e^x}{\partial x}\right)^2 dx^2} = \sqrt{(e^x)^2 dx^2} = e^x dx$$

$$S_f = e^x \cdot S_x \quad \text{OR} \quad \frac{S_f}{f} = S_x$$

~~S_f~~

CASE 5

$$f = \ln x$$

$$S_f = \sqrt{\left(\frac{\partial \ln x}{\partial x}\right)^2 dx^2} = \sqrt{\left(\frac{1}{x}\right)^2 dx^2} =$$

$$= \frac{1}{x} dx = \frac{1}{x} S_x$$

$$S_f = \frac{S_x}{x}$$

CASE 3 $\quad f = x_1 + x_2$

$$S_f = \sqrt{\left(\frac{\partial(x_1+x_2)}{\partial x_1}\right)^2 (Sx_1)^2 + \left(\frac{\partial(x_1+x_2)}{\partial x_2}\right)^2 (Sx_2)^2}$$

$$S_f = \sqrt{(Sx_1)^2 + (Sx_2)^2}$$

NOTE (THE SAME RESULT IS FOR $f = x_1 - x_2$, $(-1)^2 = 1$)

NOTE : THE SAME IS FOR $x_1 \pm x_2 \pm x_3 \pm \dots x_N$

$$S_f = \sqrt{(Sx_1)^2 + (Sx_2)^2 + \dots (Sx_N)^2}$$

CASE 2

$$f = x_1 \cdot x_2$$

$$s_f = \sqrt{\left(\frac{\partial(x_1 \cdot x_2)}{\partial x_1}\right)^2 (s_{x_1})^2 + \left(\frac{\partial(x_1 \cdot x_2)}{\partial x_2}\right)^2 (s_{x_2})^2} =$$

$$= \sqrt{(x_2)^2 (s_{x_1})^2 \underbrace{\left(\frac{x_1}{x_1}\right)^2}_{1} + (x_1)^2 (s_{x_2})^2 \underbrace{\left(\frac{x_2}{x_2}\right)^2}_{1}} =$$

$$= x_1 \cdot x_2 \sqrt{\left(\frac{s_{x_1}}{x_1}\right)^2 + \left(\frac{s_{x_2}}{x_2}\right)^2} =$$

$$s_f = f \sqrt{\left(\frac{s_{x_1}}{\bar{x}_1}\right)^2 + \left(\frac{s_{x_2}}{\bar{x}_2}\right)^2}$$

NOTE: YOU CAN DO THE SAME CALCULATION FOR $X_1 \cdot X_2$, THE RESULT IS THE SAME BECAUSE $(-)$ IS ^{2} (SEE CASE 2)

NOTE: WE EXPRESS THIS RESULT IN THE FORM

$$\frac{S_f}{f} = \underset{\text{(UNCERTAINTY)}}{\text{RELATIVE ERROR}} = \sqrt{\left(\frac{S_{x_1}}{\bar{X}_1}\right)^2 + \left(\frac{S_{x_2}}{\bar{X}_2}\right)^2}$$

NOTE: USE THE EXPRESSION FROM YOUR BOOK - IDENTIFY WHAT IS WHAT THERE

NOTE: THE SAME IS FOR $x_1, x_2, x_3 \ldots x_N$ IN ANY FUNCTION THAT CONTAINS THEIR PRODUCT OR DIVISION

$$\frac{S_f}{f} = \sqrt{\left(\frac{S_{x_1}}{X_1}\right)^2 + \left(\frac{S_{x_2}}{X_2}\right)^2 + \ldots + \left(\frac{S_{x_N}}{X_N}\right)^2}$$

CASE 1

$$f = \frac{x_1}{x_2}$$

$$S_f = \sqrt{\left(\frac{\partial\left(\frac{x_1}{x_2}\right)}{\partial x_1}\right)^2 \left(\Delta S_{x_1}\right)^2 + \left(\frac{\partial\left(\frac{x_1}{x_2}\right)}{\partial x_2}\right)^2 \left(\Delta S_{x_2}\right)^2}$$

$$= \sqrt{\left(\frac{1}{x_2}\right)^2 \left(\Delta S_{x_1}\right)^2 \cdot \underbrace{\left(\frac{x_1}{x_1}\right)^2}_{1} + \left(\frac{-x_1}{x_2^{\,2}}\right)^2 \left(\Delta S_{x_2}\right)^2}$$

$$= \left(\frac{x_1}{x_2}\right) \sqrt{\left(\frac{\Delta S_{x_1}}{x_1}\right)^2 + \left(\frac{\Delta S_{x_2}}{x_2}\right)^2} =$$

$$S_f = f \sqrt{\left(\frac{\Delta S_{x_1}}{\bar{x}_1}\right)^2 + \left(\frac{\Delta S_{x_2}}{\bar{x}_2}\right)^2}$$

NOTE : THIS IS ALSO THE RESULT/ANSWER FOR DENSITY

$$f = \rho = \frac{M}{V}$$

$$S_\rho = \rho \sqrt{\left(\frac{\Delta S_M}{\bar{M}}\right)^2 + \left(\frac{\Delta S_V}{V}\right)^2}$$

✓

WE CAN NOW SUBSTITUTE OUR "MEASURED" VALUES FOR: $X_1 \to \overline{X}_1$, and $\Delta X_1 \to S_{X_1}$

$X_2 \to \overline{X}_2$, $\Delta X_2 \to S_{X_2}$

. . . . etc (x_N)

IN OUR "NOTATION"

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 \cdot \Delta x_1^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 \Delta x_2^2} \Rightarrow$$

$$S_f = \sqrt{\ldots\ldots\ldots} \Bigg|_{x_1 = \overline{X}_1,\ \Delta x_1 = S_{x_1}\ \ldots\ \text{etc}}$$

HOW TO DO IT → DO SIMPLE DERIVATIVES:

(1) FIRST THE FUNCTION, SAY $f = \frac{x_1}{x_2}$

(like for density $\rho = \frac{M}{V}$)

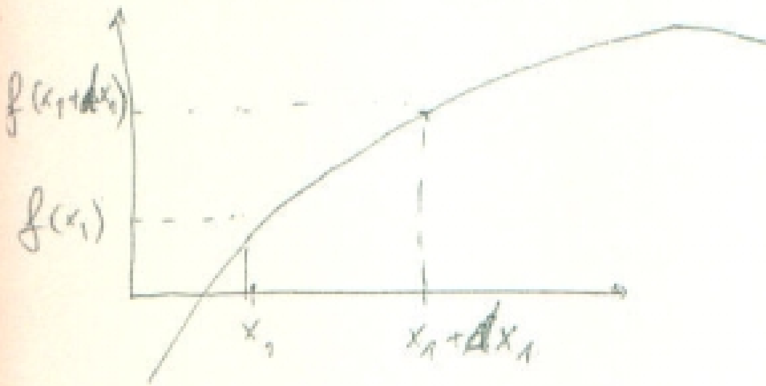
(2) SUBSTITUTE f IN THE EXPRESSION

(3) PRESENT THE RESULT IN A "USEFUL" FORM

FOR INFINITESIMAL CHANGE, FOR EACH VARIABLE X_1,

$$\frac{\partial f}{\partial x_1}$$

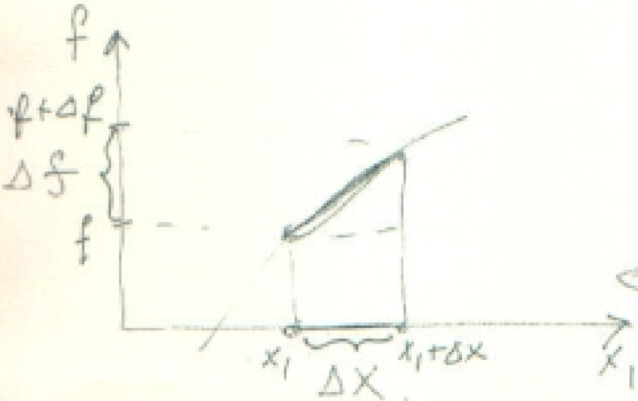
$$\frac{\partial f}{\partial x_1} = f'(x_1, x_2, \ldots) = \lim_{dx_1 \to 0} \frac{f(x_1 + dx_1) + f(x_1)}{dx_1}$$



FOR "SMALL" CHANGE OF CERTAIN VALUE WE CAN USE

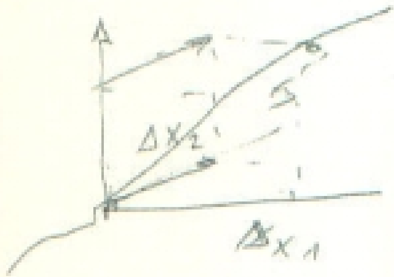
$d \to \Delta$, $dx \to \Delta x$

$df \to \Delta f$,



$$\frac{\Delta f}{\Delta x_1} = \frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x_1}$$

"PITAGORA" GEOMETRY FOR MORE THAN ONE x_i VARIABLE



$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 (\Delta x_1)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 (\Delta x_2)^2}$$

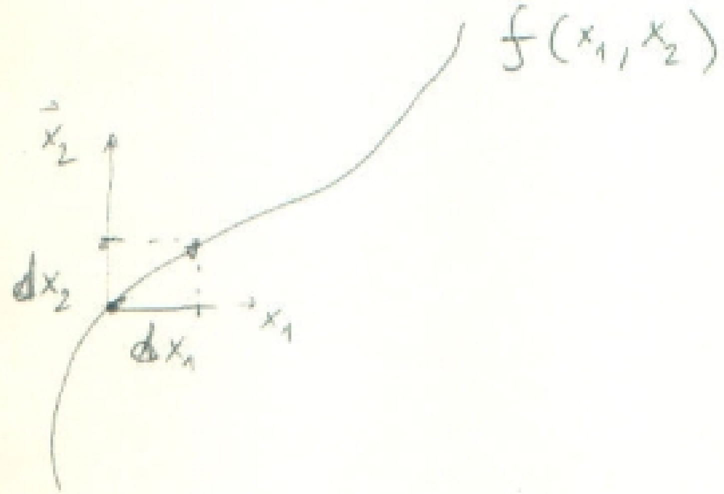
GENERAL QUESTION: WHAT IS

S_f = STANDARD DEVIATION FOR $f = f(x_1, x_2, \dots x_N)$

IF WE KNOW $\bar{x}_1$, ~~S_{x_1}~~, $\bar{x}_2$, S_{x_2} .... etc.

ANSWER ON GENERAL QUESTION:

→ WE CAN DISCUSS A 2-D EXAMPLE. IT CAN BE EXPANDED TO ANY N-DIMENSIONAL CASE AS LONG AS THE VARIABLES $x_1, \dots x_N$ ARE INDEPENDENT ("ORTHOGONAL")



FIRST, ESTABLISH HOW MUCH f DOES CHANGE IF A VARIABLE CHANGES THE FUNCITON, THAT IS HOW MUCH IS THE EFFECT OF "ERROR" IN X_1 ON THE TOTAL f

INFINITESIMAL CHANGE IS:

$$df = \frac{\partial f(x_1,x_2)}{\partial x_1} dx_1 + \frac{\partial f(x_1,x_2)}{\partial x_2} \cdot dx_2$$

PROPAGATION OF UNCERTAINTY/ERROR

GENERAL CASE: HOW TO ESTIMATE ERROR/UNCERTAINTY IN "DERIVED" FUNCTIONS, THAT IS NOT DIRECTLY MEASURED. THE FUNCTIONS CAN DEPEND ON ONE, TWO, OR MORE "DIRECTLY" MEASURED OBSERVABLES

$$f = f(x_1, x_2, x_3, \dots x_N)$$

WHERE, f IS AN ANALYTICAL ("WELL-BEHAVED") FUNCTION

EXAMPLE CASE: DENSITY ρ

$$\rho = \frac{M}{V}$$

M = mass

V = volume

FOR M we can "measure" $\overline{M}$ and s_M (standard dev.)

FOR V we can "measure" $\overline{V}$ and s_V

WHAT IS $s_\rho^{\,2}$?